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Characterizing curves satisfying the Gauss–Christoffel theorem

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ABSTRACT

In this paper we obtain the reciprocal of the classical Gauss theorem for quadrature formulas. Indeed we characterize the support of the measures having quadrature formulas with the exactness given in the Gauss theorem.

1. Introduction

Gaussian quadrature formulas is a classical subject that has been widely studied. These types of formulas allow us to compute, in an exact way, the integrals of polynomials up to a certain degree depending on the nodes and the quadrature coefficients.

The well-known Gauss theorem for the case of measures with support on the real line characterizes the nodal systems for which the quadrature formula is exact in the space of algebraic polynomials of degree less than or equal to $2m-1$, $\mathbb{P}_{2m-1}$. Indeed, if μ is a positive measure on the real line and $\{P_n(x)\}_{n\in\mathbb{N}}$ is the monic orthogonal polynomial sequence with respect to μ, then the quadrature formula $I_m(f) = \sum_{i=1}^{m} \lambda_i f(\alpha_i)$, where $\{\alpha_i\}_{i=1}^{m}$ are the zeros of $P_m(x)$, is such that $I_m(P) = \int_{supp(\mu)} P(x)\mathrm{d}\mu(x)$ for any polynomial P of degree less than or equal to $2m-1$, (see [1–3]).

In the case of measures supported on the unit circle the analogue formulas are the so-called Szegő quadrature formulas. If μ is a positive measure on the unit circle $\mathbb{T} = \{z : |z| = 1\}$, an m-point Szegő quadrature formula has the following form $I_m(f) = \sum_{j=1}^{m} A_j f(z_j)$, where the nodes belong to $\mathbb{T}$, that is, $|z_i| = 1$, $\forall i = 1, \ldots, m$ and $z_i \neq z_j$ for $i \neq j$. If $\{\Phi_n(z)\}_{n\in\mathbb{N}}$ is the monic orthogonal polynomial sequence with respect to μ and the nodes are the zeros of the para-orthogonal polynomials $\Phi_m(z) + \tau\Phi_m^*(z)$, with $|\tau| = 1$, then $I_m(P) = \int_{\mathbb{T}} P(z)\mathrm{d}\mu(z)$ for P belonging to the space of Laurent polynomials $\Lambda_{-(m-1),m-1} = span\{z^k : -(m-1) \leq k \leq m-1\}$. Moreover $\Lambda_{-(m-1),m-1}$ is the maximal domain of validity in the sense that there cannot exist any m-point quadrature formula exact for space $\Lambda_{-(m-1),m}$ or $\Lambda_{-m,m-1}$, (see [4–6]).

For another approach they can be seen the results concerning the typical extensions of an inner product given in [7,8].

If we take into account the theorem of change of variable, the preceding results can be extended to any straight line or any circumference in the complex plane. Let us see for the circumference $\mathbb{T}_1$ with centrum a and radius r that the existence of a nodal system on $\mathbb{T}$ with exactness on the space $\Lambda_{-(m-1),m-1}$ asserts the existence of another nodal system on $\mathbb{T}_1$ with

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exactness on the same space $\Lambda_{-(m-1),m-1}$. Indeed, let us consider a positive measure μ_1 on $\mathbb{T}_1$ and let us take $z = \frac{y-a}{r}$ for computing the following integral $\int_{\mathbb{T}_1} P(y)\mathrm{d}\mu_1$ for $P \in \Lambda_{-(m-1),m-1}$. Then

$$\int_{\mathbb{T}_1} P(y)\mathrm{d}\mu_1 = \int_{\mathbb{T}} P(rz+a)\frac{1}{r}\mathrm{d}\mu = \frac{1}{r}\sum_1^m A_j P(rz_j+a) = \sum_1^m \frac{A_j}{r}P(y_j).$$

Hence, in both situations, real line and unit circle, the construction of quadrature formulas needs the knowledge of the zeros of orthogonal or para-orthogonal polynomials and for increasing the exactness new computations need to be done.

In a recent contribution, (see [4]), we have studied, in a unified way, Gaussian quadrature formulas on the real line and on the unit circle. This unified approach can be extended to arbitrary measures on the complex plane $\mathbb{C}$. Thus, in the present paper we consider a finite positive Borel measure μ with infinite support on a subset S of the complex plane $\mathbb{C}$. We consider the Hilbert space $L_2(\mu)$ with the inner product $\langle f(\overline{z},z), g(\overline{z},z)\rangle = \int_S f(\overline{z},z)\overline{g(\overline{z},z)}\mathrm{d}\mu$ and we assume that $\overline{z}^j z^i \in L_2(\mu),\ \forall i,j \in \mathbb{N}\cup\{0\}$. If m is a positive integer, $m \geq 1$, we denote by $\mathbb{P}_{i,j}[\overline{z},z] = span\{\overline{z}^k z^l\ :\ 0 \leq k \leq i, 0 \leq l \leq j\}$. Following this notation we have $\mathbb{P}_{m-1}[z] = \mathbb{P}_{0,m-1}[\overline{z},z] = span\{1,\ldots,z^{m-1}\}$, and $\mathbb{P}_{m-1,0}[\overline{z},z] = span\{1,\ldots,\overline{z}^{m-1}\}$.

We recall that an m-point quadrature formula for the measure μ with nodal system $\{\alpha_i\}_{i=1}^m$ and weights $\{\lambda_i\}_{i=1}^m$ has the form $I_m(f) = \sum_{i=1}^m \lambda_i f(\alpha_i)$. In [4], following Gauss' ideas and taking into account that an m-point quadrature formula cannot be exact on $\mathbb{P}_{m,m}[\overline{z},z]$, we look for quadrature formulas with exactness in spaces of the form $\mathbb{P}_{i,j}[\overline{z},z]$ with i and j as large as possible. Next we recall the main results obtained in [4].

(1) If the measure has a nodal system with exactness on $\mathbb{P}_{m-1,m-1}[\overline{z},z]$ then the nodal system and the corresponding weights are characterized by the $m-1$ kernel of the measure. Besides we can obtain a polynomial in $\overline{z}$ and z having orthogonality properties on the support of the measure.
(2) Using the preceding characterizations we obtain, for measures supported on the line and the circle, all the nodal systems with m points and exactness on $\mathbb{P}_{m-1,m-1}[\overline{z},z]$. In particular we recover the well-known theory for the real line and we extend the classical theory for the unit circle.
(3) There exist measures such that they have not nodal systems of m points with exactness on $\mathbb{P}_{m-1,m-1}[\overline{z},z]$.

2. Characterizing supports

Our aim in this paper is to characterize the support of the measures having quadrature formulas with the exactness of Gauss theorem. So we assume that the measure μ has the following properties: (i) μ is a finite positive Borel measure with infinite support on a subset S of $\mathbb{C}$ and it has finite moments. (ii) For each $m \geq 1$ there exists a nodal system $\{\alpha_1,\ldots,\alpha_m\}$ and a system of weights $\{\lambda_1,\ldots,\lambda_m\}$ such that the quadrature formula $I_m(f) = \sum_{i=1}^m \lambda_i f(\alpha_i)$ exactly integrates $f \in \mathbb{P}_{m-1,m-1}[\overline{z},z]$. The question is: What can we say about the support of μ?

Lemma 1. *Let α_1 and α_2 be two different complex numbers. Let $A \subset \mathbb{C}$ and assume that there exists some $\theta \in [0,2\pi]$ such that for every $z \in A$ the following equality holds $(\overline{z}-\overline{\alpha_1})(z-\alpha_2) = \mathrm{e}^{\mathrm{i}\theta}(z-\alpha_1)(\overline{z}-\overline{\alpha_2})$. Then A must be a subset of a straight line or a subset of a circumference.*

Proof. (1) If $\overline{z}z$ is linearly independent with $z,\overline{z}$ and 1, then by equating the coefficients of $\overline{z}z$ on both sides of the above equality we get that $\mathrm{e}^{\mathrm{i}\theta} = 1$ and therefore $-\alpha_2\overline{z} - \overline{\alpha_1}z + \overline{\alpha_1}\alpha_2 = -\alpha_1\overline{z} - \overline{\alpha_2}z + \overline{\alpha_2}\alpha_1$ on A, or equivalently, $(\alpha_1-\alpha_2)\overline{z} - (\overline{\alpha_1}-\overline{\alpha_2})z = \overline{\alpha_2}\alpha_1 - \overline{\alpha_1}\alpha_2$, so we have $\Im[(\overline{\alpha_1}-\overline{\alpha_2})z] = -\Im(\overline{\alpha_2}\alpha_1)$. If $z = x+\mathrm{i}y$ we can rewrite this last equality as follows $\Re(\overline{\alpha_1}-\overline{\alpha_2})y + \Im(\overline{\alpha_1}-\overline{\alpha_2})x = -\Im(\overline{\alpha_2}\alpha_1)$. Since it is not possible $\Re(\overline{\alpha_1}-\overline{\alpha_2}) = \Im(\overline{\alpha_1}-\overline{\alpha_2}) = 0$, because it would imply $\alpha_1 = \alpha_2$, we obtain that z belongs to a straight line.
(2) If $\overline{z}z$ is linearly dependent with $z,\overline{z}$ and 1, then there exist $a,b,c \in \mathbb{C}$ such that $\overline{z}z = az + b\overline{z} + c$ on A, and therefore $\Im(az+b\overline{z}+c) = 0$ on A. If $z = x+\mathrm{i}y$ then $\Im[(\Re a + \mathrm{i}\Im a)(x+\mathrm{i}y) + (\Re b + \mathrm{i}\Im b)(x-\mathrm{i}y) + \Re c + \mathrm{i}\Im c] = 0$, that is, $(\Re a - \Re b)y + (\Im a + \Im b)x + \Im c = 0$. Thus we have the following two possibilities:
  (a) If $\Re a - \Re b = 0$, $\Im a + \Im b = 0$ then $\Im c = 0$ and $a = \overline{b}$ and we can write $|z|^2 = az + \overline{az} + \Re c$, that is, $2\Re(az) + \Re c = |z|^2$. Since $z = x+\mathrm{i}y$ the last expression can be written like $x^2+y^2-2x\Re a + 2y\Im a = \Re c$ and therefore z belongs to a circle.
  (b) If $\Re a - \Re b \neq 0$ or $\Im a + \Im b \neq 0$, then it is immediate to obtain that z belongs to a straight line.

Lemma 2. *Let α_1,α_2 and α_3 be three different complex numbers. For a convenient β the polynomial $((\overline{z}-\overline{\alpha_1})(z-\alpha_2)-(z-\alpha_1)(\overline{z}-\overline{\alpha_2}))(z-\alpha_3)+\beta((\overline{z}-\overline{\alpha_1})(z-\alpha_3)-(z-\alpha_1)(\overline{z}-\overline{\alpha_3}))(z-\alpha_2)$ has coefficient in z^2 equal to zero and it is a linear combination of the variables $\overline{z}z,\ \overline{z},\ z$, and 1 with some no null coefficients.*

Proof. First we assume that $\alpha_1 = 0$, $\alpha_2 = 1$, and $\alpha_3 = c$, with c an arbitrary number, $c \neq 0,1$. In this situation the polynomial becomes into $\big(\overline{z}(z-1)-z(\overline{z}-1)\big)(z-c)+\beta\big(\overline{z}(z-c)-z(\overline{z}-\overline{c})\big)(z-1)$. Since the coefficient of z^2 must be zero then $\beta = \frac{-1}{\overline{c}}$. After doing some calculations the polynomial can be written as $(z-\overline{z})(z-c) - \frac{1}{\overline{c}}(-\overline{c}z + c\overline{z})(z-1) = z\overline{z}\left(-1+\frac{c}{\overline{c}}\right) + \overline{z}\left(c - \frac{c}{\overline{c}}\right) + z(1-c)$, that is, it is a linear combination of $\overline{z}z,\ \overline{z},\ z$, and 1. Let us consider now that α_1,α_2 and α_3 are arbitrary and different complex numbers. We do the change of variable $y = \frac{z-\alpha_1}{\alpha_2-\alpha_1}$ which transforms α_1, α_2, and α_3

into 0, 1, and $c \neq 0, 1$ respectively. In the same way $z-\alpha_1$, $z-\alpha_2$, and $z-\alpha_3$ become into $(\alpha_2-\alpha_1)y$, $(\alpha_2-\alpha_1)(y-1)$, and $(\alpha_2-\alpha_1)(y-c)$. Therefore, if the polynomial would be a linear combination of $\bar{z}z$, $\bar{z}$, z, and 1 with null coefficients, then the transformed expression in $\bar{y}y$, $\bar{y}$, y, and 1 will have null coefficients, which is impossible.

Theorem 3. *Let $I_m(f) = \sum_{i=1}^{m} \lambda_i f(\alpha_i)$ be a quadrature formula, based on a nodal system $\{\alpha_1, \ldots, \alpha_m\}$, with $m \geq 3$, and such that it exactly integrates $f \in \mathbb{P}_{m-1,m-1}[\bar{z}, z]$. Then the support of the measure μ is, up to a finite number of points, a straight line or a circumference.*

Proof. Let us take $m \geq 3$ and the quadrature formula $I_m(f)$, based on the nodes $\{\alpha_1, \ldots, \alpha_m\}$. We consider $p_k(z) = (\bar{z}-\overline{\alpha_k}) \prod_{i=1,\, i\neq k}^{3}(z-\alpha_i) \prod_{i=4}^{m}(z-\alpha_i)$ for $k = 1, 2, 3$. It is clear that $p_1(z), p_2(z), p_3(z) \in \bar{z}\mathbb{P}_{m-1}[z] + \mathbb{P}_{m-1}[z]$. We distinguish the following two cases:

(1) We assume that on the support of the measure the following property holds: the variable $\bar{z}z^{m-1}$ is a linear combination of the variables $\bar{z}z^{m-2}, \ldots, \bar{z}, z^{m-1}, \ldots, 1$. In this case we consider the polynomials $p_1(z)$ and $p_2(z)$ and we distinguish the following two subcases:

(a) We assume that the expansion of $p_1(z)$ in the basis obtained from $\{\bar{z}z^{m-2}, \ldots, \bar{z}, z^{m-1}, \ldots, 1\}$ has null term in z^{m-1}. In this case $|p_1(z)|^2$ belongs to $\mathbb{P}_{m-1,m-1}[\bar{z}, z]$ and we can compute $\|p_1\|$ using the quadrature formula $\|p_1(z)\|_\mu^2 = \int_{\mathbb{C}} p_1(z)\overline{p_1(z)}\mathrm{d}\mu = I_m\left(p_1(z)\overline{p_1(z)}\right) = 0$. This last equality implies that $p_1(z) = 0$ on the support and therefore the support is a finite set.

(b) We assume that the expansions of $p_1(z)$ and $p_2(z)$ in the basis obtained from $\{\bar{z}z^{m-2}, \ldots, \bar{z}, z^{m-1}, \ldots, 1\}$ are such that the coefficients of z^{m-1} are different from 0. In this situation, for a convenient β, the polynomial $p_1(z)-\beta p_2(z)$ has null term in z^{m-1}, it is such that $|p_1(z)-\beta p_2(z)|^2 \in \mathbb{P}_{m-1,m-1}$ and its norm can be computed using the quadrature formula obtaining

$$\begin{aligned}\|p_1(z)-\beta p_2(z)\|_\mu^2 &= \int_{\mathbb{C}} (p_1(z)-\beta p_2(z))(\overline{p_1(z)-\beta p_2(z)})\mathrm{d}\mu \\ &= I_m\left((p_1(z)-\beta p_2(z))(\overline{p_1(z)-\beta p_2(z)})\right) = 0.\end{aligned}$$

This implies the following equality on the support $p_1(z) = \beta p_2(z)$. Since both polynomials have the same norm, then $\beta = \mathrm{e}^{\mathrm{i}\theta}$ for some $\theta \in [0, 2\pi]$. Therefore on the support it holds that either $\prod_3^m (z-\alpha_i) = 0$ (a finite set of points) or $(\bar{z}-\overline{\alpha_1})(z-\alpha_2) - \mathrm{e}^{\mathrm{i}\theta}(z-\alpha_1)(\bar{z}-\overline{\alpha_2}) = 0$. This last equation leads to a straight line or a circle taking into account Lemma 1. In summary, in this case the support is, up to a finite set of points, a subset of a straight line or a circle.

(2) Now we assume that on the support of the measure the variable $\bar{z}z^{m-1}$ is not a linear combination of the variables $\bar{z}z^{m-2}, \ldots, \bar{z}, z^{m-1}, \ldots, 1$. Then the functions $q_1(z) = p_1(z) - p_2(z)$ and $q_2(z) = p_1(z) - p_3(z)$ have null term in $z^{m-1}\bar{z}$. Thus we consider the following two subcases:

(a) If, for example, $q_1(z)$ has null term in z^{m-1} then $q_1(z) \in \bar{z}\mathbb{P}_{m-2}[z] + \mathbb{P}_{m-2}[z]$ and we can compute its norm using the quadrature formula obtaining $\|q_1(z)\|^2 = I_m\left((p_1(z)-p_2(z))\overline{(p_1(z)-p_2(z))}\right) = 0$. Therefore it holds that $q_1(z) = 0$ on the support, that is, $\prod_3^m (z-\alpha_i) = 0$ and hence the support is a finite set or $(\bar{z}-\overline{\alpha_1})(z-\alpha_2) = (z-\alpha_1)(\bar{z}-\overline{\alpha_2})$, which leads to a straight line (see the proof of Lemma 1 for the details). Thus in this case the support is, up to a finite set of points, a straight line.

(b) If both functions $q_1(z)$ and $q_2(z)$ have terms in z^{m-1} then for some $\beta \in \mathbb{C}$ we get that $r(z) = q_1(z) - \beta q_2(z)$ is a polynomial without components neither in $\bar{z}z^{m-1}$ nor in z^{m-1}. We compute its norm like in the preceding cases obtaining that it is zero. This implies that on the support it must be $r(z) = 0$, that is:

$$\begin{aligned} 0 = \big(p_1(z)-p_2(z)\big) - \beta\big(p_1(z)-p_3(z)\big) = \Big\{&((\bar{z}-\overline{\alpha_1})(z-\alpha_2) - (z-\alpha_1)(\bar{z}-\overline{\alpha_2}))\,(z-\alpha_3) \\ &- \beta\,((\bar{z}-\overline{\alpha_1})(z-\alpha_3) - (z-\alpha_1)(\bar{z}-\overline{\alpha_3}))\,(z-\alpha_2)\Big\} \prod_4^m (z-\alpha_i).\end{aligned}$$

This leads to a finite number of points or to the fact that the equation of Lemma 2 holds on the support. From Lemma 2 the first member is a linear combination of the variables $z\bar{z}$, z, $\bar{z}$, and 1 with no null coefficients. Since $z\bar{z}$ cannot be a linear combination of z, $\bar{z}$ and 1 ($\bar{z}z^{m-1}$ is linearly independent of $\bar{z}\mathbb{P}_{m-2}[z] + \mathbb{P}_{m-1}[z]$), then the linear combination equal to zero with no null coefficients would be with the variables z, $\bar{z}$ and 1. Therefore there exist constants a, b, and c such that $az + b\bar{z} + c = 0$ on the support. If $z = x + \mathrm{i}y$ then the last equation gives $(\Re(a)+\Re(b))x + (-\Im(a)+Im(b))y + \Re(c) = 0$ and $(\Im(a)+\Im(b))x + (\Re(a)-\Re(b))y + \Im(c) = 0$. The system cannot have all the coefficients zero and since it has solution, then it must be a point or a line. So in this case we have again a finite set of points or a line and a set of isolated points.

3. The reciprocal of the classical Gauss theorem

Under the assumption that the measure μ has the following property: For each $m \geq 3$ there exists a nodal system such that the corresponding quadrature formula exactly integrates functions belonging to $\mathbb{P}_{m-1,m-1}[\bar{z}, z]$, in the preceding section we have proved that the support of the measure is: (i) A straight line and a set of isolated points. Besides, the isolated points belong to the nodal system, or (ii) A circle and a set of isolated points. Besides, the isolated points belong to the nodal system.

Next we are going to improve Theorem 3 excluding the isolated points. We begin with the case of the line. Let us assume that μ is a measure supported on $\mathbb{R} \cup \{\alpha_1, \ldots, \alpha_k\}$ where $\{\alpha_1, \ldots, \alpha_k\}$ are isolated points outside $\mathbb{R}$ with masses $\{\mu_1, \ldots, \mu_k\}$. Let us also assume that $\{\alpha_1, \ldots, \alpha_k\} \cup \{\alpha_{k+1}, \ldots, \alpha_m\}$, with $m - k \geq 3$, is a nodal system with weights $\{\lambda_1, \ldots, \lambda_m\}$, that exactly integrates $f \in \mathbb{P}_{m-1,m-1}[\overline{z}, z]$. In this case we can write $\int_{\mathbb{C}} f(z, \overline{z}) \mathrm{d}\mu(z) = \int_{\mathbb{R}} f(z, \overline{z}) \mathrm{d}\mu(z) + \sum_1^k \mu_i f(\alpha_i, \overline{\alpha_i})$. Using the exactness of the quadrature formula we get $\int_{\mathbb{C}} f(z, \overline{z}) \mathrm{d}\mu(z) = \sum_1^m \lambda_i f(\alpha_i, \overline{\alpha_i})$ and therefore we obtain $\int_{\mathbb{R}} f(z, \overline{z}) \mathrm{d}\nu(z) = \sum_1^k (\lambda_i - \mu_i) f(\alpha_i, \overline{\alpha_i}) + \sum_{k+1}^m \lambda_i f(\alpha_i, \overline{\alpha_i})$, where this formula is exact in $\mathbb{P}_{m-1,m-1}[\overline{z}, z]$. Since we know (see [4]) that in this situation the nodes must be real numbers, this case must be excluded. Using the theorem of change of variable the same result can be obtained for arbitrary lines in the complex plane. Similar arguments can be used for the case of a circumference.

Finally we are in conditions to state the reciprocal of Gauss theorem.

Theorem 4. *Let μ be a finite positive Borel measure with arbitrary support on $\mathbb{C}$, such that for each positive integer m there exists an m-point quadrature formula which exactly integrates functions in $\mathbb{P}_{m,m-1}[\overline{z}, z]$. Then the support of the measure μ is a straight line.*

Proof. Applying the preceding results we get that the support is a line or a circle. Taking into account that in the case of the circle we have not exactness for functions belonging to $\mathbb{P}_{m,m-1}[\overline{z}, z]$, we deduce that the support of the measure μ is a line.

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